Chapter 49

MBS VALUATION AND PREPAYMENTS

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Abstract

This paper not only provides a comparison of recent models in the valuation of mortgage-backed securities but also proposes an integrated model that addresses important issues of path-dependence, exogenous prepayment, transaction costs, mortgagors' heterogeneity, and the housing devaluation effect.

Recent research can be categorized into two frameworks: empirical and theoretical option pricing. Purely empirically derived models often consider estimation of the prepayment model and pricing of the mortgage-backed security as distinct problems, and thus preclude explanation and prediction for the price behavior of the security. Some earlier theoretical models regard mortgage-backed securities as default-free callable bonds, prohibiting the mortgagors from exercising the default (put) option, and therefore induce bias on the pricing of mortgage-backed securities. Other earlier models assume homogeneity of mortgagors and consequently fail to address important issues of premium burnout effect and the path-dependence problem.

The model proposed is a two-factor model in which the housing price process is incorporated to account for the effect of mortgagor's default and to capture the impact of housing devaluation. Default is correctly modeled in terms of its actual payoff through a guarantee to the investors of the security such that the discrepancy is eliminated by assuming mortgage securities as either default-free or uninsured. Housing prices have been rising at unsustainable rates nation wide, especially along the coasts, suggesting a possible substantial weakening in house appreciation at some point in the future. The effect of housing devaluation is specifically modeled by considering the possibility that the mortgagor might be restrained from prepayment even if interest rates make it advantageous to refinance.

Mortgagors' heterogeneity and the separation of exogenous and endogenous prepayments are explicitly handled in the model. Heterogeneity is incorporated by introducing heterogeneous refinancing transaction costs. The inclusion of heterogeneous transaction costs not only captures premium burnout effect but also solves the path-dependence problem. Finally, the model separates exogenous prepayment from endogenous prepayment, and estimates their distinct magnitudes from observed prepayment data. This construction provides a better understanding for these two important components of prepayment behavior. The generalized method of moments is proposed and can be employed to produce appropriate parameter estimates.

Keywords: MBS valuation; option pricing theory; exogenous and endogenous prepayments; housing devaluation effect; devaluation trap; transaction costs of refinancing and default; generalized method of moments; path dependency; premium burnout effect; heterogeneity

49.1. Introduction

The main objective of this paper is to gain a better understanding of the valuation of mortgage-backed securities. Mortgage-backed securities have attracted unprecedented investor interest over the last decade, spurring tremendous growth in the market for this important financial instrument. There are over $7.7 trillion worth of residential mortgage loans outstanding, an amount far exceeding the size of the corporate debt market. Approximately $5.1 trillion worth of securitized mortgage-backed securities and CMOs are outstanding, and well over $1.8 trillion new mortgage-backed securities and whole loans pools are issued each year for the past three years.[1] Mortgage-backed securities are extensively held by every class of institutional investor, including commercial banks, saving institutions, insurance companies, mutual funds, and pension plans.

An in-depth study of the valuation of mortgage-backed securities is of interest to financial economists because mortgage-backed securities have unique characteristics that are distinct from other contingent claims, such as monthly amortization, negative convexity, premium burnout, and path-dependence. This paper examines recent developments in the area of valuing mortgage-backed securities and proposes a model that accommodates these factors affecting the price of mortgage-backed securities.

The core issue in valuing mortgage-backed securities is the modeling of the prepayment behavior of mortgagors in the pool backing the security. Continuous-time option pricing methodology has been a popular method in the mortgage-backed securities valuation because of the obvious parallel between the call option and the right of a mortgagor to prepay. In order to model the mortgagors' prepayment behavior more realistically, recent theoretical models have added modifications to the original stock option pricing theory framework. The first of these modifications broadly accounts for prepayment due to reasons exogenous to financial consideration, such as moving and job changes. The second group of modification addresses transaction costs. The third considers heterogeneity among mortgagors, and the fourth group discusses the separation of exogenous prepayment and endogenous prepayment.

The observation that homeowners clearly do not prepay as objectively as option pricing models imply has motivated many researchers to add prepayment functions that allow prepayments for reasons that are exogenous to purely financial considerations. Such research includes the work of Dunn and McConnell (1981a,b), and Brennan and Schwartz (1985), and most of the prepayment functions have been arbitrary. The main drawback of adding an arbitrary prepayment function is that it does not aid in the identification of the factors responsible for prepayment behavior. Identifying these factors would go a long way toward enhancing the explanatory power of the model.

Applying the option pricing theory to the valuation of residential mortgage-backed securities, one can see a departure from the perfect market assumption when homeowners face transaction costs upon refinancing or defaulting. For this reason, Dunn and Spatt (1986) and Timmis (1985) add homogenous refinancing transaction costs in their models to adjust the prepayment speeds from those implied in the frictionless economic environment. Kau et al. (1993) also add the transaction cost of default in their modeling of the probability of default for residential mortgages.

Addressing mortgagors' heterogeneity is a more complex matter. Many earlier models assumed homogeneity among mortgagors to avoid complexity in the pricing process. However, the assumption of mortgagors' homogeneity fails to address the issue of premium burnout which is an important empirical effect of homeowner heterogeneity. And this assumption also results in a path-dependent problem when numerically solving the optimal refinancing strategies backwards. The premium burnout effect is the tendency of prepayments from premium pools to slow down over time, with all else held constant. If a large number

of mortgagors have already prepaid, those remaining are likely to have a relatively low probability of prepaying. Conversely, the smaller the number of previous prepayments, the higher the probability of prepaying by the remaining mortgagors. The aforementioned path-dependent problem occurs because any mortgage pool contains a group of mortgagors who behave differently in their prepayment decisions: these mortgagors differ in their willingness or ability to prepay their loans under favorable circumstances. As a result, without knowing either the type of mortgagor or the entire path of interest rates from origination, backward optimization is not applicable because there is no way of knowing whether the earlier prepayment exercise is optimal.

Johnston and Van Drunen (1988), and Davidson et al. (1988) improve on the homogenous transaction cost model by introducing heterogeneous transaction models. They assume that different homeowners face different levels of refinancing transaction costs. In addition to the ability to capture the premium burnout, the inclusion of heterogeneous transaction costs also solves the path-dependent problem encountered when pooling individual mortgagors, who behave differently in their prepayment decisions.

Another common problem in existing models is the lack of differentiation between exogenous prepayment and endogenous prepayment. This lack of distinction between the two thereby precludes explanation of the interrelation between these important behavioral components. Endogenous prepayment refers to any prepayment decision that occurs in response to changes in underlying economic processes, such as the interest rate. Stanton (1990) incorporates an endogenous decision parameter that enables separate estimations of endogenous prepayment and prepayment for exogenous reasons. As a result, the explanatory power of the model is improved. In addition to the inclusion of the previously discussed modifications, our model introduces two adjustments. One is the treatment of mortgagors' right to default in the content of mortgage-backed securities valuation. And the other is the impact of the housing prices on prepayment behavior.

Although default has been modeled as a put option in the models of residential mortgages or commercial mortgage-backed securities, many earlier models have not incorporated it in the valuation of residential mortgage-backed securities. This is because government agency guarantees lead to the perception that securities are default-free. Default should be taken into consideration because there is a payoff difference between a guaranteed mortgage-backed security and a default-free security. The payoff from a guarantee in the event of default is the par amount rather than the market value of the security, thus producing an asymmetric return for investors.

In modeling default, we expand previous default-free models into a default-risky model in which the housing price process is included as a second-state variable. Default is explicitly modeled in terms of its actual payoff through a guarantee to the investors of the residential mortgage-backed security. This is in contrast to models for individual mortgages or commercial mortgages in which mortgages are neither insured nor guaranteed. Consequently, the payoff in the event of default in these cases is the value of the house. By correctly modeling the effect of default, our model reduces the discrepancy from assuming mortgage-backed securities as either default-free or uninsured.

The housing price process is incorporated in the model not only to account for the effect of default on security price, but also to determine its impact on the prepayment behavior of mortgagors. The effect of housing prices on prepayment is specifically modeled by considering the possibility that the mortgagor might be restrained from prepaying even if interest rates make it advantageous to refinance. This is because housing prices have fallen to the extent that the mortgagor is no longer qualified for refinancing.

The model we propose not only captures the fundamental characteristics of the mortgagors' prepayment behavior but it also combines parametric heterogeneity and variability of the decision

parameter to the extent that our model can come closer than previous models in describing empirical prepayment behaviors.

49.2. The Model

The central issue in valuing mortgage-backed securities is the treatment of prepayment uncertainty. The valuation model of mortgage-backed securities proposed here is based on the continuous-time option pricing methodology. This methodology treats the right of a mortgagor to prepay as a call option and the right to default as a put option. Modifications to the assumption of perfect capital markets and the principle that borrowers act to minimize the market cost of their mortgages are required to portray mortgagors' actual prepayment behavior in a more realistic manner.

According to Dunn and McConnell (1981) and Brennan and Schwartz (1985), we allow mortgagors to prepay for reasons exogenous to purely financial considerations. In contrast to their models that assume arbitrary exogenous prepayment functions, our model utilizes the proportional hazard function and can be estimated from observable prepayment data.

To account for the fact that homeowners face transaction costs when they prepay or default on their mortgages, we follow Johnston and Van Drunen (1988). Consequently, we add heterogeneous refinancing transaction costs in our models to adjust the prepayment speeds from those implied in the frictionless economic environment. Following Kau et al. (1993), we also add the transaction cost of default in modeling the effect of default.

Default has been modeled as a put option in the valuation of residential mortgages or commercial mortgage-backed securities. However, many models have not incorporated default in the valuation of residential mortgage-backed securities because government agency guarantees lead to the perception that securities are default-free. Moreover, there is a significant difference between the payoff of a guaranteed mortgage-backed security and that of a default-free security. The payoff from insurance in the event of default is the par amount rather than the market value of the security, producing an asymmetric return for investors.

Kau and associates (1992) develop a two-factor model for both prepayment and default only in the context of evaluating individual mortgages, where mortgages are considered as uninsured. As discussed in the Chapter 49, the payoff from uninsured mortgages is the value of the house when the mortgage is defaulted. In our model, the payoff to the investor from default is explicitly modeled as insured mortgages. This eliminates the potential bias in the pricing of mortgage-backed securities.

A significant relationship between observed prepayment and housing prices data pointed out by Richard (1991) leads us a final adjustment of the two-factor model. The housing price process is brought in not only to account for the effect of default on security price, but also to determine its restraining effect on mortgagors' refinancing decisions.

Figure 49.1 outlines these differences between one-and two-factor models and the innovations presented in this study.

In the one-factor model, the prepayment decision responds to the level of interest rates. The two-factor model adds two additional termination outcomes that follow from the level of housing prices. At very low housing prices, the mortgagors may default regardless of the interest rate in order to cut their losses. Finally, the mortgagor might be restrained from prepaying even if interest rates make it advantageous to refinance. This occurs when the housing prices fall to the extent that the new loan cannot cover the costs of refinancing.

In addition to capturing these fundamental characteristics of the mortgagor's termination behavior, this model aggregates the underlying pool of mortgages according to the heterogeneity of transaction costs. And it is the specification of heterogeneous transaction costs that also solves the path-dependent problem displayed by pooled mortgages.

The following first section pertains to the modeling of termination decisions affected by exogenous

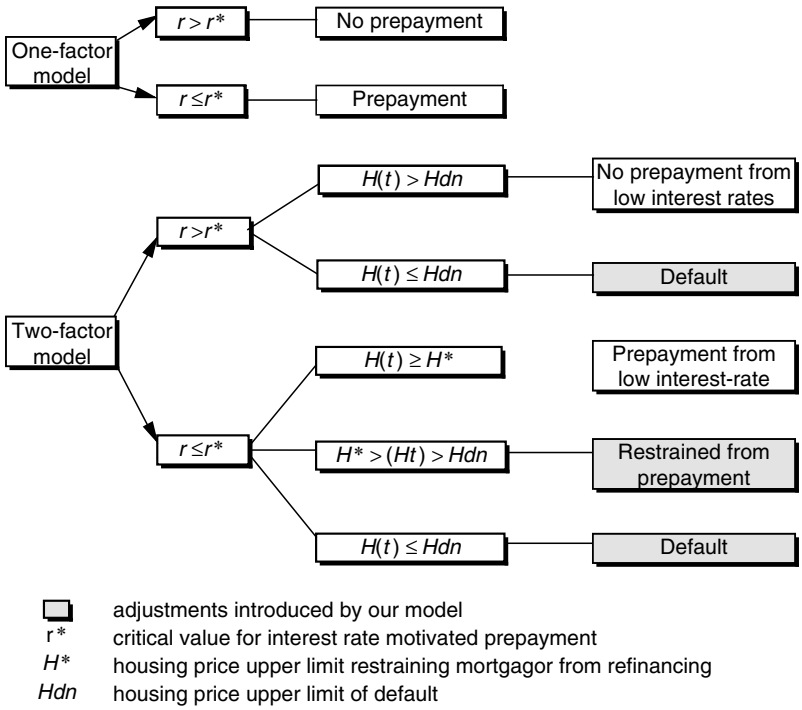

Figure 49.1. Model trees

and endogenous factors, housing prices, and transaction costs. The later section introduces our model, which is a two-factor pricing framework that provides exact security prices given underlying interest rate and housing prices processes, and precludes arbitrage opportunities.

49.2.1. Modeling Issues

49.2.1.1. Exogenous Prepayment

In practice, exogenous reasons for termination include factors such as relocation, death, divorce, or natural disasters. Exogenous prepayments are also known as turnover prepayments. A hazard function is used to model exogenous prepayment as follows:

$$\pi(t) = \lim_{\delta t \to 0^+} \frac{\Pr(\text{Exogeneous prepayment in}(t, t+\delta t) | \text{ No prepayment prior to } t)}{\delta t}. \tag{49.1}$$

There are numerous parametric methods used in the analysis of duration data and in the modeling of aging or failure processes. We use the exponential distribution in the model for its simplicity. The distribution is characterized by the constant hazard function

$$\pi(t) = \pi, t \geq 0 \text{ and } \pi > 0. \tag{49.2}$$

The probability that an individual has not prepaid for exogenous reasons until time t is given by the survival function $S(t)$,

$$S(t) = \mathrm{e}^{-\pi(t)} = \mathrm{e}^{-\pi t}, t \geq 0 \tag{49.3}$$

49.2.1.2. Endogenous Termination

A mortgage is terminated when mortgagors either prepay or default on their mortgages. Any termination which affects the cash flows passed through to the investors will have an impact on the price of the mortgage-backed securities. Throughout the model, endogenous termination is defined as any rational termination decision that occurs in response to underlying economic processes rather than personal considerations.

We assume that mortgagors maximize their current wealth, or equivalently, minimize their liabilities. Mortgagors' liabilities can be thought of as

composed of three parts. The first part consists of owing the scheduled streams of cash flows associated with the mortgage. The second part constitutes their option to prepay at any time, which is equivalent to possessing a call option. And the third part consists of mortgagor's option to default, which functions as a put option. Option pricing theory is, therefore, an appropriate method for determining the value of mortgagors' mortgage liability.

A model of mortgage pricing should incorporate both refinancing transaction costs and default transaction costs in order to more accurately portray the decision-making processes of mortgagors. Although including transaction costs causes the resulting termination strategy to deviate from the perfect market assumption, the strategy still remains rational.

In order to derive the magnitude of endogenously determined termination, we follow Stanton (1990) and introduce ρ, which measures the frequency of mortgagors' termination decisions. The time between successive decision points is described as an exponential distribution. If we let T_i be one such decision point, and T_{i+1} the next, then

$$\Pr(T_{i+1} - T_i > t) = \mathrm{e}^{-\rho t} \tag{49.4}$$

If mortgagors are continually re-evaluating their decisions, then the parameter ρ takes on a value of infinity. If mortgagors never make endogenous termination decisions and only terminate for exogenous reasons, then ρ takes on a value of zero. If ρ takes on a value between these limits, then this signifies that decisions are made at discrete times, separated on average by $1/\rho$.

Given this specification, the magnitude of endogenized termination can be estimated and studied. The contribution of this device is to separate the magnitude of endogenized termination from that of exogenous termination. It also serves to help understand the actual termination behavior of mortgagors. Without this specification, it would be difficult to know the proportion of termination from endogenous optimization decisions and the proportion due to exogenous factors.

Utilizing the definitions from Sections 49.2.1.1 and 49.2.1.2, we notice that the optimal exercise strategy immediately leads to a statistical representation of the time to terminate for a single mortgagor. If termination is due exclusively to exogenous factors, then the termination rate is π and the survival function is defined as in Equation (49.4). When termination occurs for endogenous reasons, the probability that the mortgagor terminates in a small time interval, δt, is the probability that the mortgagor neither prepays for exogenous reasons nor makes a rational exercise decision during this period. This survival function can be approximated by

$$S(t) = \begin{cases} \mathrm{e}^{-\pi\delta t} \cdot \mathrm{e}^{-\rho\delta t} = \mathrm{e}^{-(\pi+\rho)\delta t} & \text{if endogenous termination} \\ \mathrm{e}^{-\pi\delta t} & \text{if no endogenous termination} \end{cases}. \tag{49.5}$$

49.2.1.3. Transaction Costs and Aggregation of Heterogeneous Mortgages

The cash flows that accrue to the investor of a mortgage-backed security are not determined by the termination behavior of a single mortgagor, but by that of many mortgagors within a pool. To cope with the path-dependent problem caused by the heterogeneity within a pool of mortgages, we assume that the different refinancing transaction costs each mortgagor faces is the only source of heterogeneity. Although the costs of initiating a loan vary among different types of mortgages, some of the most common costs borrowers face include credit report, appraisal, survey charges, title and recording fees, proration of taxes or assessments, hazard insurance, and discount points.

The transaction costs of individual mortgagors are drawn from a univariate discrete distribution, which allows for underlying heterogeneity in the valuation of the mortgage-backed security. A better way to choose the underlying distribution that represents this heterogeneity would be to look at summary statistics of transaction costs actually incurred by mortgagors when they refinanced. A discrete rectangular distribution is chosen for

its simplicity and the task of determining which distribution improves the fit is left for future research.

The value of the security is equal to the expected value of the pool of mortgages weighted by the proportions of different refinancing transaction cost categories. Suppose that each X_i (the refinancing transaction costs faced by mortgagor i) is drawn from a discrete rectangular, or uniform distribution

$$\Pr(x = a + ih) = M^{-1}, \quad i = 1, \ldots, M \tag{49.6}$$

Various standard forms are in use. For this application, we set $a = 0$, $h = RM^{-1}$, so that the values taken by x are RM^{-1}, $2RM^{-1}$, $\ldots$, R. The upper bound R of the transaction cost is set at 10 percent. The distribution for the transaction costs is then defined as:

$$\Pr\left(x = i\frac{0.1}{M}\right) = M^{-1}, \quad i = 1, \ldots, M \tag{49.7}$$

In principle, given any initial distribution of transaction costs, it is possible to value a mortgage-backed security backed by a heterogeneous pool of mortgages in a manner similar to the valuation of a single mortgage. If the value of individual mortgages is known, then the value of the pool is the sum of these individual values. When the value of individual mortgages is not known, but a distribution of transaction costs is generated that accounts for heterogeneity, the expected value of a pool of mortgages is the sum of the transaction cost groups times the probability of their occurrence in the pool.

Recall from Section 49.1.2 that for a given transaction cost X_i and state of the world, if any mortgagor finds it optimal to terminate, the hazard rate is the sum of the exogenous prepayment rate, π, and the endogenized termination rate, ρ. If it is not optimal to terminate, the hazard rate falls back to the background exogenous prepayment rate π.

Models that neither permit the estimation of ρ nor consider exogenous factors in the prepayment decision imply that $\rho = \infty$ and $\pi = 0$, and the single-transaction cost level predicts that all mortgages will prepay simultaneously. Adding heterogeneous transaction costs addresses the problem of path dependence, however, keeping the same parameter values still does not permit hesitation in the prepayment decision. Although prepayment rates fluctuate, in reality, they do tend to move fairly smoothly. The effect of setting ρ to a value other than ∞ is to permit a delay even when it is optimal to prepay. And prepayment need not occur at all if interest rates or housing prices change such that it is no longer optimal. The actual value of ρ determines how fast this drop occurs. Thus, combining parametric heterogeneity and variability of the parameter ρ would allow the model to come closer than previous rational models to describe empirical prepayment behavior.

49.2.2. A Model for Pricing Mortgage-Backed Securities

49.2.2.1. Termination Decision of a Single Mortgagor

The following is a model of rational prepayment behavior of mortgages that extends the rational prepayment models of Stanton (1990) and Kau and associates (1993). Mortgagors may terminate their mortgages for endogenous financial reasons that include interest rates and housing prices, or for exogenous reasons. They also face transaction costs, which are used to differentiate mortgagors and solve the path-dependent problem. Mortgagors choose the strategy that minimizes the market value of the mortgage liability.

The following assumptions are employed:

1. Trading takes place continuously and there are no taxes or informational asymmetries.
2. The term structure is fully specified by the instantaneous riskless rate $r(t)$. Its dynamics are given by.

$$\mathrm{d}r = \kappa(\mu_{\mathrm{r}} - r)\mathrm{d}t + \sigma_{\mathrm{r}}\sqrt{r}\mathrm{d}z_{\mathrm{r}} \tag{49.8}$$

3. The process to capture the housing price is assumed to follow a Constant Elasticity of Variance (CEV) diffusion process

$$\mathrm{d}H = \mu_{\mathrm{H}} H \mathrm{d}t + \sigma_{\mathrm{H}} H^{\gamma/2} \mathrm{d}z_{\mathrm{H}}, \tag{49.9}$$

where $\mu_{\mathrm{H}}, \sigma_{\mathrm{H}} > 0$, $0 < \gamma < 2$, and $\{z_{\mathrm{H}}(t), t \geq 0\}$ is a standard Wiener Process, which may be correlated with the process $\{z_{\mathrm{r}}(t), t \geq 0\}$ When $\gamma = 2$, the process is lognormal.

The underlying state variables in the model are the interest rate $r(t)$ and the housing price $H(t)$. By applying the arbitrage argument, the value of the ith mortgage liability $V^i(r,H,t)$ satisfies the following partial differential equation:

$$\begin{aligned}&\frac{1}{2}\sigma_{\mathrm{r}}^2 r V^i_{\mathrm{rr}} + \rho\sigma_{\mathrm{r}}\sigma_{\mathrm{H}}\sqrt{r}H^{\gamma/2}V^i_{\mathrm{rH}} \\ &\quad + \frac{1}{2}\sigma_{\mathrm{H}}^2 H^{\gamma} V^i_{\mathrm{HH}} + [\kappa(\mu_{\mathrm{r}} - r) - \lambda r]V^i_{\mathrm{r}} \\ &\quad + rHV^i_{\mathrm{H}} + V^i_{\mathrm{t}} - rV^i \\ &\quad = 0,\end{aligned} \tag{49.10}$$

where λr represents factor risk premium.

The value of the mortgage liability is also required to satisfy the following boundary conditions:

1. At maturity T, the value of a monthly amortization bond is equal to the monthly payment:

$$V^i(r,\ H,\ T) = MP$$

2. As r approaches infinity, the payoff of the underlying mortgage bond approaches zero:

$$\lim_{r\to\infty} V^i(r,H,t) = 0$$

Figure 49.2 summarizes the remaining conditions, which establish the boundaries of the various circumstances affecting the termination decision.

3. At any time t, the mortgage value satisfies the following conditions:

Let $V^i(r,H,t^+) = V^i(r,H,t+1) + MP$, then

$$V^i(r,H,t) = \begin{cases} V^i(r,H,t^+) & \text{if } H(t) > Hdn \text{ and } U(t)(1+X_i) > V_i\ (r,\ H,\ t^+) \text{ if continued} \\ U(t) & \text{if } H(t) > Hdn \text{ and } V^i(r,H,t^+) \geq U\ (t)\ (1+Xi\ ) \text{ if refinanced} \\ U(t) & \text{if } H(t) \leq Hdn \text{ if defaulted} \end{cases}$$

where $U(t)$ is the principal remaining at time t. Hdn is the boundary of default, defined as the housing price times the cost of default, or $Hdn = (V^i r, H, t^+)/(1+d).X_i$ is the prepayment transaction costs for individual i and d is the transaction cost of default for all individuals. This boundary condition defines the default and refinancing regions in Figure 49.2. When housing prices fall so low that they are exceeded by the default cost-adjusted mortgage value, the mortgagor will exercise their put option by defaulting. The refinancing region describes a situation in which

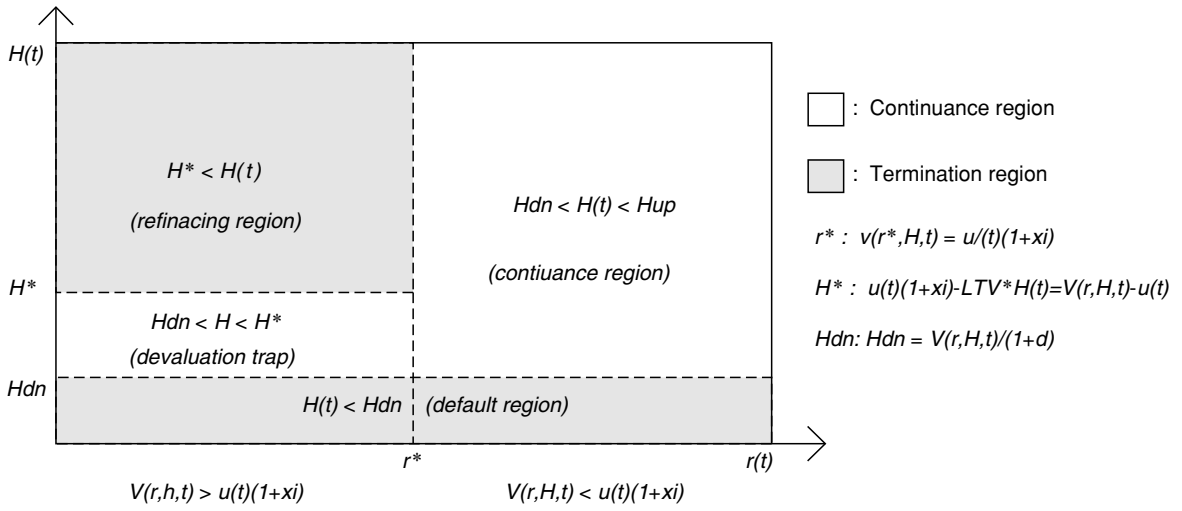

Figure 49.2. Diagram of boundary conditions

the interest rate falls to the point where the mortgage value is greater than the refinancing cost-adjusted unpaid principal. In this case, the mortgagor exercises the call option by refinancing their loan. The value of the mortgage liability takes on the value of unpaid principal $U(t)$ unadjusted by transaction costs $(1 + X_i)$, because the refinancing costs are collected by the third party who services the mortgage.

4. To improve on the previous model, we have included the effect of housing prices on the termination decision

$$V^i(r, H, t) = V^i(r, H, t^+) \text{if} H^* > H(t) > Hdn \text{ and } V^i(r, H, t^+) \geq U(t) (1 + Xi) \text{ if restrained,}$$

where LTV is the loan-to-value ratio and H^* is determined at

$$U(t) + (1 + Xi) - LTV * H(t) = V^i(r, H, t) - U(t). \quad (49.11)$$

This condition encompasses the devaluation trap. The devaluation trap occurs when housing prices fall between H^* and Hdn, where the costs of refinancing exceed its benefits. The mortgagor will be unable to refinance their loan, even though interest rates are advantageous, because they will have to pay the difference out of their pocket. And since the housing price remains above the default threshold, the mortgagor continues the mortgage. The present value of costs is determined by the left-hand side of Equation (49.11), that is the difference between the unpaid principal plus refinancing transaction cost and the new loan amount, which is the housing price times the loan-to-value ratio. The benefit of refinancing is given by the right-hand side of Equation (49.11), i.e. the mortgage value minus the unpaid principal. The role of the loan-to-value ratio is important in determining the size of the devaluation trap. The higher loan-to-value ratios result in decreases in the range of the devaluation trap.

Working back one month at a time, we can value the ith mortgage liability $V^i(r, H, t)$ by solving Equation (49.10), given boundary condition 1 through 4. Given π and ρ, we can also calculate the probability that the mortgage is terminated in month t. Denote P_e the probability of termination if only exogenous prepayment occurs. Denote P_r the probability of termination if it is endogenous conditions that lead to a decision to terminate in month t. According to the survival function Equation (49.5), these termination probabilities are given by

$$P_r = 1 - e^{-(\pi+\rho)/12} \text{ if endogenous termination}$$
$$P_e = 1 - e^{-\pi/12} \text{ if no endogenous termination}$$

We can now calculate the expected value of a single mortgage liability. That is

$$V^i(r, H, t) = \begin{cases} (1 - P_r)V^i(r, H, t^+) + P_r U(t) \\ \text{(if endogenous termination)} \\ (1 - P_e)^i(r, H, t^+) + P_e U(t) \\ \text{(if no endogenous termination)} \end{cases}$$

49.2.2.2. Valuation of a Pool of Mortgages

To determine the value of the mortgage-backed security at any time t, as mentioned above, we can simply take the expected value of pooled mortgage liabilities

$$V(r, H, t) = \sum_{i=1}^{M} V^i(r, H, t) \times P(X_i = x) \quad x \in (0, 0.1] \quad (49.12)$$

49.3. Estimation

A model for valuing mortgage-backed securities was described that permits the determination of the security's price for given parameter values describing exogenous and endogenous factors that contribute to the termination decision. The next logical step would be to estimate these parameter values from prepayment data. In this sector, the generalized method-of-moment technique is

proposed for the estimation, where the termination probability at any given time t is required for equating the population and sample moments. In order to accomplish this, we must determine the model in terms of the probability rather than in terms of the dollar value of the security.

49.3.1. Determination of the Expected Termination Probability

In addition to equating the population and sample moments when the generalized method-of-moment technique is employed for the estimation, the calculation of termination probability is useful because it can also be utilized to determine the expected cash flows for any other mortgage-related securities, such as collateralized mortgage obligations. We first restate the procedure for determining the price in order to provide a comparison to the procedure for determing termination probability.

49.3.1.1. Procedure for Determining the Security Price

In this model, the uncertain economic environment a homeowner faces is described by two variables: the interest rate and the housing price. The term structure of the interest rate is assumed to be generated from the stochastic process described in Equation (49.8) and the process of the housing prices is represented in Equation (49.9). Assuming perfect capital markets, the present value $V^i(r, H, t)$ of the mortgage contract at time t is of the form

$$V^i(r, H, t) = \tilde{E}_t\left[\mathrm{e}^{-\int_t^T r(t)\mathrm{d}t}\bar{V}^i(T)\right], \tag{49.13}$$

where $\bar{V}^i(T)$ is the terminal value of the mortgage liability at expiration date T. This equation states that the value of the mortgage is equivalent to the discounted-expected-terminal payoff under the risk-neutral measure. By Girsanov's theorem, under certain circumstances, the change in measure merely produces a change in drift in the underlying diffusions. Consequently, one must substitute the risk-adjusted processes for the actual stochastic processes in Equations (49.8) and (49.9), which in this case are

$$\mathrm{d}r = (\kappa\mu_{\mathrm{r}} - (\kappa + \lambda)r)\mathrm{d}t + \sigma_{\mathrm{r}}\mathrm{d}\tilde{z}_{\mathrm{r}} \tag{49.14}$$

and

$$\mathrm{d}H = rH\mathrm{d}t + \sigma_{\mathrm{H}}\mathrm{d}\tilde{z}_{\mathrm{H}}. \tag{49.15}$$

When the housing price process is transformed to its risk-adjusted form, the actual required rate of return on the house μH drops out of the equation. Therefore μH does not influence the mortgage and default option values. We know that the mortgage value $V^i(r, H, t)$ satisfies the partial differential equation specified in Equation (49.10). And thus, with the appropriate terminal and boundary conditions, the value of the mortgage is determined by solving this partial differential equation (PDE) backwards in time.

49.3.1.2. Deriving the Expected Termination Probability of Mortgage i

In order to implement the parameter estimation, we are now concerned with the actual occurrence of termination instead of the dollar value of the mortgage. We begin the derivation of termination probability with the following definition:

$$\begin{aligned} P^i(r, H, t) = \Pr(&(r(\tau), H(\tau), \tau) \\ &\in \text{termination region of mortgage } i, \text{ for some} \\ &\tau > t, \text{ given}(r(t), H(t), t) = (r, H, t)) \end{aligned} \tag{16}$$

where (r, H, t) are the interest rate and housing price at current time t, while $P^i(r, H, t)$ is the probability that termination ever occurs beyond the current situation. The general theory of stochastic processes allows that such a probability satisfies the Kolmogorov backward equation

$$\begin{aligned} &\frac{1}{2}\sigma_r^2 r P^i_{rr} + \rho\sigma_r\sigma_H\sqrt{r}H^{\gamma/2}P^i_{rH} + \frac{1}{2}\sigma_H^2 H^{\gamma}P^i_{HH} \\ &\quad + \kappa(\mu_r - r)P^i_r + \mu_H P^i_H + P^i_t = 0 \end{aligned} \tag{17}$$

To describe the boundary and terminal conditions, we denote Ω for the part of (r,H,t) space outside the termination region, while $\partial\Omega$ forms the termination boundary. Using this notation, we have the terminal and boundary conditions

$$P^i(r,H,T) = \begin{cases} 1/M & \text{if } (r,H,t) \in \partial\Omega, t \in (o,\ T) \\ 0 & \text{otherwise} \end{cases} \tag{49.18}$$

These conditions merely state the obvious principle that termination has a probability of $1/M$ if the conditions lead to a decision to terminate and has a probability of zero if the mortgage continues. One might recall that a pool of the mortgagors is segregated according to a discrete uniform distribution with M groups. Hence, if the environment is within the termination region, the probability of termination of any given mortgage group is $1/M$, rather than one.

The determination of probability of termination does not involve discounting, and as such the nonhomogeneous rV^i term from Equation (49.10) is excluded from Equation (49.17). Considering we are concerned with the actual incidence of termination and not the dollar value of the termination option, the real process in Equations (49.8) and (49.9) are used for $r(t)$ and $H(t)$ rather than the risk-adjusted processes. Therefore, μH is required when the probability of termination is calculated although μH has no effect on the dollar value of mortgage liability. Solving the valuation problem from Equation (49.10) gives the index result of the termination region $\partial\Omega$, which consequently enters into the terminal conditions of Equation (49.17). These conditions are treated as a fixed-boundary problem in the solving of Equation (49.17), rather than as a free-boundary problem as in the solving of Equation (49.10).

Solving Equation (49.17) subject to the boundary conditions in Equation (49.18) yields the termination probability at any grid $(r, H, t,\ X_i)$ for the individual mortgage liability i, called P_{it}. Recalling that the hazard rate for mortgage i from Equation (49.5) takes on a value of π if it is not optimal to terminate for endogenous reasons, and takes on a value of $(\pi + \rho)$ otherwise, and thus the expected termination probability for mortgage i, P^*_{it}, is calculated as

$$P^*_{it} = P_e(1 - P_{it}) + P_r P_{it}. \tag{5.19}$$

49.3.1.3. Determination of the Expected Termination Level of Pool j

Since the distribution of the refinancing transaction cost X_i is independent of the underlying stochastic processes, the expected termination for any given pool j is calculated by counting the proportion of terminations in each transaction cost group. If we denote P^*_{jt} as the expected termination probability for a given pool j, then

$$\begin{aligned} P^*_{jt} &= \sum_{i=1}^{M} P^*_{it} = \sum_{i=1}^{M} P_e(1 - P_{it}) + P_r P_{it} \\ &= P_e\left(1 - \sum_{i=1}^{M} P_{it}\right) + P_r \sum_{i=1}^{M} P_{it} \end{aligned} \tag{49.20}$$

Equation (49.20) permits us to calculate the expected termination probability for a given pool j without having to calculate the expected probability of each individual mortgage i.

49.3.2. Estimation Approach

49.3.2.1 Generalized Method of Moments (GMM)

The generalized method of moments procedure set out by Hansen (1982) and Hansen and Singleton (1982) has been widely used in financial market applications and in labor market application. The procedure is a limited-information method analogous to two-stage least squares. GMM provides a means of estimating parameters in a model by matching theoretical moments of the data, as a function of the parameters, to their sample counterparts.

The usual way of proceeding is to identify error functions of the parameters and observable data which have an expectation of zero, conditional on the information available at the time the data are

observed. That is, if we let θ_0 denote the true vector of parameter values, there are error functions $e_{it}(\theta_0)$, $i = 1, 2, \ldots, M$, satisfying the orthogonality conditions

$$E[e_{it}(\theta_0)|I_t] = 0, \tag{49.21}$$

where e_{it} is a function of the parameters and of data up to and including time t and I_t is the information set at time t. This equation states that these error functions have a mean zero conditional on the information set at time t, when the functions are evaluated at the true parameter value. The implication is that the errors must be uncorrelated with the variables in the information set I_t, and thus, if z_{jt} is a finite dimensional vector of random variables that are I_t measurable, then by the law of iterated expectations

$$E[e_{it}(\theta_0)z_{jt}] = 0. \tag{49.22}$$

If e is a $M \times T$ matrix, and z is $N \times T$, this can be rewritten as

$$E[g_t(\theta_0)] = 0, \tag{49.23}$$

where g is the $MN \times T$ matrix formed by taking the direct product of e and z. This equation is the basis for the GMM technique. Suppose the sample counterpart of this set of population moments is the MN-vector valued function $g_T(\theta)$, which is defined by

$$g_T(\theta)_i = \frac{1}{T}\sum_{t=1}^{T} g_{it}(\theta). \tag{49.24}$$

The usual GMM estimation procedure involves minimizing a quadratic form of the type

$$Q_T(\theta) = g_T(\theta)' W g_T(\theta), \tag{49.25}$$

where W is some positive-definite weighting matrix. This is usually done in two steps. First, take W to be the identity matrix and perform one minimization. Next calculate W_T, the sample estimator of

$$W_0 = (E[g_t(\theta_0)g_t(\theta_0)'])^{-1}, \tag{49.26}$$

and use this as the weighting matrix in the second stage. As long as $W_T \to W_0$ almost surely, then the asymptotic variance matrix of the GMM estimator is

$$\sum_0 = \frac{1}{T}[(E[\partial g_t(\theta_0)'/\partial\theta])\, W_0 (E[\partial g_t(\theta_0)/\partial\theta'])]^{-1} \tag{49.27}$$

In addition, the statistic $TQ_T(\hat{\theta})$, which is sample size times the minimized value of the objective function, $g_T(\theta)' W g_T(\theta)$, is distributed as a chi-squared random variable with degrees of freedom equal to the dimension of $g_t(\theta 0)$ less the number of estimated parameters. This statistic provides a test of the over-identifying restrictions.

49.3.2.2. Moment Restrictions

The typical moment condition to use is the expectation of the difference between the observed prepayment level and its expected value, defined appropriately, equal zero. If we denote φ_{it} for the proportion of pool i prepaying in month t. The expected value of φ_{it}, t conditional on the information set at time t follows from above, assuming that the distribution of transaction costs among the mortgages remaining in the pool is known. If the termination probability of pool i at time t is P^*_{it}, then

$$E[\varphi_{it}|I_t] = P_e(1 - P^*_{it}) + P_r P^*_{it}, \tag{49.28}$$

where $P_r = 1 - e^{-(\pi+\rho)\delta t}$, $P_e = 1 - e^{-\pi\delta t}$ are previously defined and P^*_{it} is calculated from the previous section.

It is possible to calculate unconditional moment conditions by multiplying these conditional moment conditions in Equation (49.28) and appropriate elements of the information set. Define the residual for pool i at time t as

$$e_{it} = \varphi_{it} - \bar{\varphi}_{it}. \tag{49.29}$$

This satisfies the following expression:

$$E[e_{it}(\theta_0)|I_t'] = 0, \tag{5.30}$$

where I_t' is a subset of the full information set at time t, which includes the interest rate path. Given

this, it is possible to create more moment conditions as above. If z_{jt} is an element of I'_t, then $E[e_{it}z_{jt}] = 0$. However, z_{jt} may not be any variable that gives information about the actual sequence of prepayments. For example, setting z_{jt} equal to a lagged value of the prepayment level is not valid because the expected residual may be correlated with lagged prepayment levels. The implication of this is that there will be positive serial correlation in the residuals e_{it}. Hence, if the residuals are stacked in the usual way, averaging across time periods, one will have to deal with this serial correlation in calculating the appropriate standard errors for the GMM estimators.

To avoid the issue of serial correlation, the residuals can be stacked by averaging across pools, instead. Under the null hypothesis of independent pools, this way of stacking will result in no correlation between the contemporaneous residuals from different pools. Therefore, by assuming that the mortgages are drawn from a well-behaved underlying distribution, the sample estimator W_T is still a consistent estimator of the optimal weighting matrix W_0, and the usual asymptotic standard error results are valid.

49.4. Conclusion

The valuation model of mortgage-backed securities proposed here is a model that extends the rational option pricing approach used by previous authors. This model is able to capture many important empirical regularities observed in prepayment behavior that have previously been modeled successfully using only purely empirically derived prepayment models. However, in these purely empirical models, estimation of the prepayment behavior and the valuation of a mortgage-backed security are often treated as completely separate problems. This model prevents *ad hoc* integration of the estimation of prepayment and the valuation of a mortgage-backed security, and links these two into a structured model. Therefore, this model can address economic questions that are beyond the scope of purely empirical models, while possessing a simple reduced form representation that allows estimation using observed prepayment data.

This integrated model captures the fundamental characteristics of a mortgage-backed security, such as exogenous prepayment, endogenous prepayment, transaction costs of refinancing and default, heterogeneity among mortgagors, and the issue of path dependence. In addition, the treatment of embedded options, the prepayment (call) option and the default (put) option, are modeled with care to accommodate more realistic aspects of a mortgagor's behavior. In particular, the payoff from the incident of default is modeled as an insured mortgage, so that the potential discrepancy in the pricing of the mortgage backed security is eliminated.

Another important innovation of the model is the explicit modeling of the housing devaluation effect that was the prevailing phenomenon in the early 1990s due to declined home prices. Over last several years, housing prices have been rising at unprecedented rates. A correction in the housing market is likely to occur in the near future and trigger a devaluation-induced prepayment slowdown. The term used for the devaluation effect is "devaluation trap" because the effect is activated only when housing prices fall to a degree at which the costs of refinancing exceed its benefits. The mortgagors are trapped and unable to refinance their loans even though interest rates are advantageous, because the new loans entitled from devaluated houses are no longer sufficient to cover the costs of refinancing.

Two constituents allow this model to come closer than previous models in describing empirical prepayment and price behavior. These are the incorporation of mortgagors' heterogeneity and the delaying of the rational prepayment decisions of mortgage holders. The heterogeneity of mortgagors is accomplished by introducing heterogeneous refinancing transaction costs. And the mortgagors' prepayment decisions are assumed to occur at discrete intervals rather than continuously, as was

assumed with previous rational models. Hence, these two combined factors produce smoother prepayment behavior as observed in the actual data, and allow the model to generate prices that exceed par without requiring excessive transaction costs.

It is known that utilizing maximum likelihood as a means of estimating the parameters in parametric hazard models of prepayment is problematic, given the constitution of available prepayment data. Thus, by utilizing an alternative approach, the generalized method of moments, the model parameters can be estimated. This approach overcomes the problems associated with maximum likelihood in this setting.

NOTE

1. Taken from the Federal Reserve Bulletin, Inside MBS &ABS, and UBS.

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